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Non-local PDEs with a state-dependent delay term presented by Stieltjes integral [1]

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Parabolic partial differential equations with state-dependent delays (SDDs) are investigated. The delay term presented by Stieltjes integral simultaneously includes discrete and distributed SDDs. The singular Lebesgue-Stieltjes measure is also admissible. The conditions for the corresponding initial value problem to be well-posed are presented. The existence of a compact global attractor is proved.

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1 Introduction

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We investigate parabolic partial differential equations (PDEs) with delay. Studying of this type of equations is based on the well-developed theories of the ordinary differential equations (ODEs) with delays [11, 7, 1] and PDEs without delays [8, 9, 15, 14]. Under certain assumptions both types of equations describe a kind of dynamical systems that are infinite-dimensional, see [2, 31, 6] and references therein; see also [32, 4, 5, 3] and to the monograph [37] that are close to our work.

In many evolution systems arising in applications the presented delays are frequently *state-dependent* (SDDs). The theory of such equations, especially the ODEs, is rapidly developing and many deep results have been obtained up to now (see e.g. [33, 34, 35, 17, 19] and also the survey paper [12] for details and references).

The PDEs with state-dependent delays were first studied in [22, 13, 23]. An alternative approach to the PDEs with discrete SDDs is proposed in [25]. This approach is based on the so-called *ignoring condition* [25]. Approaches to equations with discrete and distributed SDDs are different. Even in the case of ODEs, the discrete SDD essentially complicates the study since, in general, the corresponding nonlinearity is not locally Lipschitz continuous on open subsets of the space of continuous functions, and familiar results on existence, uniqueness, and dependence of solutions on initial data and parameters from, say [11, 7] fail (see [36] for an example of the non-uniqueness and [12] for more details).

In this work, in contrast to previous investigations, we consider a model where two different types of SDDs (discrete and distributed) are presented simultaneously (by Stieltjes integral). The singular Lebesgue-Stieltjes measure is also admissible. Moreover, all the assumptions on the delay (see A1-A5 below) allow the dynamics when along a solution the number and values of discrete SDDs may change, the whole discrete and/or distributed delays may vanish, disappear and appear again. This property allows us to study models where some subsets of the phase space are described by equations with purely discrete SDDs, others by equations with purely distributed SDDs and there are subsets which need the general (combined) type of the delay. A solution could be in different subsets at

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different time moments. This property particularly means that not only the values of the delays are state-dependent, but the *type* of the delay is *state-dependent* as well. We study mild solutions and their asymptotic properties (the existence of an attractor is proved). The results could be applied to the diffusive Nicholson's blowflies equation with SDDs.

2 The model with state-dependent delay and basic properties

Consider the following non-local partial differential equation with a state-dependent delay term F presented by Stieltjes integral

$$\frac{\partial}{\partial t}u(t,x) + Au(t,x) + du(t,x) = \big(F(u_t)\big)(x), \tag{1}$$

with

$$\big(F(u_t)\big)(x) \equiv \int_{-r}^{0} \left\{ \int_{\Omega} b\left(u(t+\theta,y)\right) f(x-y)dy \right\} \cdot dg(\theta,u_t), \quad x\in\Omega, \tag{2}$$

where A is a densely-defined self-adjoint positive linear operator with domain $D(A)\subset L^2(\Omega)$ and compact resolvent, which means that $A : D(A) \to L^2(\Omega)$ generates an analytic semigroup, $\Omega \subset \mathbb{R}^{n_0}$ is a smooth bounded domain, $f : \Omega - \Omega \to R$ is a bounded measurable function, $b : \mathbb{R} \to \mathbb{R}$ stands for a locally Lipschitz map, $d \in \mathbb{R}, d \geq 0$, and the function $g : [-r,0]\times C([-r,0];L^2(\Omega)) \to [0,r] \subset \mathbb{R}_+$ denotes *a state-dependent delay.* Let $C \equiv C([-r,0];L^2(\Omega))$. Norms defined on $L^2(\Omega)$ and C are denoted by $||\cdot||$ and $||\cdot||_C$, respectively, and $\langle\cdot,\cdot\rangle$ stands for the inner product in $L^2(\Omega)$. As usual for delay equations, we denote $u_t \equiv u_t(\theta) \equiv u(t+\theta)$ for $\theta\in[-r,0]$.

We consider equation (1) with the initial condition

$$u|_{[-r,0]} = \varphi \in C \equiv C([-r,0];L^2(\Omega)). \tag{3}$$

We assume the following.

A1.) *For any $\varphi\in C$, the function $g : [-r,0]\times C([-r,0];L^2(\Omega)) \to \mathbb{R}$ is of bounded variation on $[-r,0]$. The variation $V^0_{-r}g$ of g is* `uniformly bounded` *i.e.*

$$\exists M_{Vg} > 0 \,:\, \forall\varphi\in C \quad\Rightarrow\quad V^0_{-r}g(\varphi) \leq M_{Vg}. \tag{4}$$

It is well-known that any Lebesgue-Stieltjes measure (associated with g) may be split into a sum of three measures: discrete, absolutely continuous and singular ones. We will denote the corresponding splitting of g as follows

$$g(\theta,\varphi) = g_d(\theta,\varphi) + g_{ac}(\theta,\varphi) + g_s(\theta,\varphi), \tag{5}$$

where $g_d(\theta,\varphi)$ is a step-function, $g_{ac}(\theta,\varphi)$ is absolutely continuous and $g_s(\theta,\varphi)$ is singular continuous functions (see [16] for more details). We will also denote the continuous part by $g_c \equiv g_{ac}+g_s$.

Now we assume

A2.) *For any $\theta\in[-r,0]$, the functions g_{ac} and g_s are continuous with respect to their second coordinates i.e. $\forall\theta\in[-r,0] \quad \forall\varphi^n,\varphi\in C : ||\varphi^n-\varphi||_C \to 0\,(n\to+\infty) \Rightarrow g_{ac}(\theta,\varphi^n) \to g_{ac}(\theta,\varphi)$ and $g_s(\theta,\varphi^n)\to g_s(\theta,\varphi)$.*

Remark 1. *We notice that a discrete state-dependent delay does not satisfy assumption A2). More precisely, we may consider the discrete SDD* $\eta : C \to [0,r]$ *which is presented by the step-function* $g(\theta,\varphi) = 0$ *for* $\theta \in [-r, -\eta(\varphi)]$ *and* $g(\theta,\varphi) = 1$ *for* $\theta \in (-\eta(\varphi), 0]$. *It is easy to see that for any sequence* $\{\varphi^n\} \subset C$, *such that* $\eta(\varphi^n) \to \eta(\varphi)$ *and* $\eta(\varphi^n) > \eta(\varphi)$ *one has for the value* $\theta_0 = -\eta(\varphi)$ *that* $g(\theta_0, \varphi^n) \equiv 1 \neq 0 \equiv g(\theta_0, \varphi)$, *i.e. A2) does not hold.*

A3.) *The step-function* $g_d(\theta,\varphi)$ *is continuous with respect to its second coordinate in the sense that discontinuities of* $g_d(\theta,\varphi)$ *at points* $\{\theta_k\} \subset [-r, 0]$ *satisfy the property: there are* `continuous` *functions* $\eta_k : C \to [0,r]$ *and* $h_k : C \to R$ *such that* $\theta_k = -\eta_k(\varphi)$ *and* $h_k(\varphi)$ *is the jump of* g_d *at point* $\theta_k = -\eta_k(\varphi)$ *i.e* $h_k(\varphi) \equiv g_d(\theta_k + 0, \varphi) - g_d(\theta_k - 0, \varphi)$.

Taking into account that g_d *may, in general, have infinite number of points of discontinuity* $\{\theta_k\}$, *we assume that the series* $\sum_k h_k(\varphi)$ *converges* `absolutely` *and* `uniformly on any bounded subsets of C`.

Remark 2. *Following notations of (5), we conclude that A3 means that for any* $\chi \in C$ *one has* $\Phi_d(\chi) \equiv \int_{-r}^{0} \chi(\theta)\, dg_d(\theta,\varphi) = \sum_k \chi(\theta_k) \cdot h_k(\varphi) = \sum_k \chi(-\eta_k(\varphi)) \cdot h_k(\varphi)$.

Lemma 1. *Assume the function* b *is a Lipschitz map* ($|b(s) - b(t)| \leq L_b |s-t|$), *satisfying* $|b(s)| \leq C_1|s| + C_2, \forall s \in \mathbb{R}$ *with* $C_i \geq 0$ *and* f *is measurable and bounded* ($|f(x)| \leq M_f$). *Under assumptions A1)- A3), the nonlinear mapping* $F : C \to L^2(\Omega)$, *defined by (2), is continuous.*

Remark 3. *We emphasize that nonlinear map* F *is not Lipschitz in the presence of discrete SDDs (i.e. when* $g \neq g_c$*). The proof of lemma 1 is based on the properties of the uniformly convergent series and the first Helly's theorem [16, page 359].*

Proof of lemma 1. We first split our g on continuous $g_c \equiv g_{ac} + g_s$ and discontinuous g_d parts (see (5)). This splitting gives the corresponding splitting of $F = F_c + F_d$, where F_c corresponds to the continuous part $g_c \equiv g_{ac} + g_s$.

$\Diamond$ Let us first consider the *part* F_c

We write

$$F_c(\varphi) - F_c(\psi) = I_1 + I_2, \tag{6}$$

where we denote

$$I_1 = I_1(x) \equiv \int_{-r}^{0} \left\{ \int_\Omega [b(\varphi(\theta,y)) - b(\psi(\theta,y))]\, f(x-y)\, dy \right\} dg_c(\theta,\varphi), \tag{7}$$

$$I_2 = I_2(x) \equiv \int_{-r}^{0} \left\{ \int_\Omega b(\psi(\theta,y)) f(x-y)\, dy \right\} d\,[g_c(\theta,\varphi) - g_c(\theta,\psi)], \quad x \in \Omega. \tag{8}$$

One can check that

$$||I_1|| \leq L_b M_f |\Omega| \cdot ||\varphi - \psi||_C \cdot V_{-r}^{0} g(\varphi). \tag{9}$$

This estimate and A1 show that $||I_1|| \to 0$ when $||\varphi - \psi||_C \to 0$. To show that $||I_2|| \to 0$ when $||\varphi - \psi||_C \to 0$ we use assumptions A1 and A2 to apply the first Helly's theorem [16, page 359].

$\Diamond$ Now we prove the continuity of F_d (*discrete delays*). Let us fix any $\varphi \in C$ and consider a sequence $\{\varphi^n\} \subset C$ such that $||\varphi^n - \varphi||_C \to 0$ when $n \to \infty$. Our goal is to prove that $||F_d(\varphi^n) - F_d(\varphi)|| \to 0$.

Following the notations of A3 (see also remark 2), we write

$$F_d(\varphi) = F_d(\varphi)(x) = \sum_k \int_\Omega b(\varphi(-\eta_k(\varphi), y))f(x-y)dy \cdot h_k(\varphi)$$

and split as follows

$$F_d(\varphi^n) - F_d(\varphi) \equiv K_1^n + K_2^n + K_3^n,$$

where

$$K_1^n = K_1^n(x) \equiv \sum_k \int_\Omega \left[b(\varphi^n(-\eta_k(\varphi^n), y)) - b(\varphi(-\eta_k(\varphi^n), y))\right] f(x-y)dy \cdot h_k(\varphi^n),$$

$$K_2^n = K_2^n(x) \equiv \sum_k \int_\Omega b(\varphi(-\eta_k(\varphi^n), y))f(x-y)dy \cdot \left[h_k(\varphi^n) - h_k(\varphi)\right],$$

$$K_3^n = K_3^n(x) \equiv \sum_k \int_\Omega \left[b(\varphi(-\eta_k(\varphi^n), y)) - b(\varphi(-\eta_k(\varphi), y))\right] f(x-y)dy \cdot h_k(\varphi).$$

Using the Lipschitz property of b one may check that

$$||K_1^n|| \leq L_b M_f |\Omega|^{3/2} ||\varphi^n - \varphi||_C \cdot \sum_k |h_k(\varphi^n)|. \tag{10}$$

Now we discuss K_2^n. The grough condition of b implies $|b(\varphi(-\eta_k(\varphi^n), y))f(x-y)| \leq (C_1|\varphi(-\eta_k(\varphi^n), y)| + C_2)M_f$. Hence $|\int_\Omega b(\varphi(-\eta_k(\varphi^n), y))f(x-y)dy| \leq C_1 M_f \int_\Omega |\varphi(-\eta_k(\varphi^n), y)|dy + C_2 M_f|\Omega| \leq M_f(C_1|\Omega|^{1/2}||\varphi||_C + C_2|\Omega|)$. Here we used the Cauchy-Schwartz inequality for $\int_\Omega |\varphi(-\eta_k(\varphi^n), y)|dy \leq ||\varphi(-\eta_k(\varphi^n))|| \cdot |\Omega|^{1/2} \leq ||\varphi||_C \cdot |\Omega|^{1/2}$. One sees that

$$|K_2^n(x)| \leq M_f(C_1|\Omega|^{1/2}||\varphi||_C + C_2|\Omega|) \sum_k |h_k(\varphi^n) - h_k(\varphi)|.$$

Since the right-hand side of the last estimate is independent of x, we get

$$||K_2^n|| \leq M_f(C_1|\Omega| \cdot ||\varphi||_C + C_2|\Omega|^{3/2}) \sum_k |h_k(\varphi^n) - h_k(\varphi)|. \tag{11}$$

In a similar way we obtain

$$||K_3^n|| \leq M_f L_b |\Omega| \sum_k |h_k(\varphi)| \cdot ||\varphi(-\eta_k(\varphi^n)) - \varphi(-\eta_k(\varphi)||. \tag{12}$$

Now we should explain why $||K_j^n|| \to 0$ as $n \to \infty$ for $j = 1, 2, 3$. The first property $||K_1^n|| \to 0$ follows from A3 and (10). In (11), the series converges uniformly with respect to n since the condition $||\varphi^n - \varphi||_C \to 0$ implies that $\{\varphi, \varphi^n\}$ is a bounded subset of C. Assumption A3 guarantees that each $|h_k(\varphi^n) - h_k(\varphi)|$ is continuous with respect to φ^n and tends to zero when $n \to \infty$. Due to the uniform convergence we arrive at $||K_2^n|| \to 0$. To show that $||K_3^n|| \to 0$ we also mention that each $|h_k(\varphi)| \cdot ||\varphi(-\eta_k(\varphi^n)) - \varphi(-\eta_k(\varphi)||$ (see (12)) is continuous with respect to φ^n and tends to zero when $n \to \infty$ due to A3 and the continuity of $\varphi \in C$. The uniform convergence (w.r.t. φ^n) of the series in (12) follows from the estimate $|h_k(\varphi)| \cdot ||\varphi(-\eta_k(\varphi^n)) - \varphi(-\eta_k(\varphi)|| \leq |h_k(\varphi)| \cdot 2||\varphi||_C$ (the right-hand side is independent of n!) and the Weierstrass dominant (uniform) convergence theorem. We conclude that $||K_3^n|| \to 0$. Since all $||K_j^n|| \to 0$ as $n \to \infty$ for $j = 1, 2, 3$ we proved the property $||F_d(\varphi^n) - F_d(\varphi)|| \to 0$. The proof of lemma 1 is complete. ∎

3 Mild solutions

In our study we use the standard

Definition 1. *A function $u \in C([-r,T]; L^2(\Omega))$ is called a* `mild solution` *on $[-r,T]$ of the initial value problem (1), (3) if it satisfies (3) and*

$$u(t) = e^{-At}\varphi(0) + \int_0^t e^{-A(t-s)} \{F(u_s) - d \cdot u(s)\}\, ds, \quad t \in [0,T]. \tag{13}$$

Theorem 1. *Under assumptions of lemma 1, initial value problem (1), (3) possesses a mild solution for any $\varphi \in C$.*

The existence of a mild solution is a consequence of the continuity of $F : C \to L^2(\Omega)$, given by lemma 1, which gives us the possibility to use the standard method based on the Schauder fixed point theorem (see e.g. [37, theorem 2.1, p.46]). The solution is also global (is defined for all $t \geq -r$), see e.g. [37, theorem 2.3, p. 49].

To get the uniqueness of mild solutions we need the following additional assumptions.

A4.) *The total variation of function $g_c \equiv g_{ac} + g_s$ satisfies the* `Lipschitz condition`

$$V_{-r}^0[g_c(\cdot,\varphi) - g_c(\cdot,\psi)] \leq L_{Vg_c}||\varphi - \psi||_C. \tag{14}$$

A5.) *Discrete generating function g_d satisfies the* `uniform ignoring condition` *i.e.*

- $\exists \eta_{ign} > 0$ *such that* `all` η_k *and* h_k *"*`ignore`*" values of $\varphi(\theta)$ for $\theta \in (-\eta_{ign}, 0]$ i.e.*

$$\exists\, \eta_{ign} > 0 : \forall \varphi^1, \varphi^2 \in C : \forall \theta \in [-r, -\eta_{ign}], \Rightarrow \varphi^1(\theta) = \varphi^2(\theta) \quad \Longrightarrow$$

$$\eta_k(\varphi^1) = \eta_k(\varphi^2),\, h_k(\varphi^1) = h_k\varphi^2).$$

Remark 4. *Assumption A5 is the natural generalization to the case of multiple discrete state-dependent delays of the ignoring condition introduced in [25]. For more details and examples see [25].*

Theorem 2. *Assume the function b is a Lipschitz map ($|b(s) - b(t)| \leq L_b|s-t|$), satisfying $|b(s)| \leq M_b$, $\forall s \in R$ and f is measurable and bounded ($|f(x)| \leq M_f$). Under assumptions A1)- A5), initial value problem (1), (3) possesses a unique mild solution for any $\varphi \in C$. The solution is continuous with respect to initial data i.e. $||\varphi^n - \varphi||_C \to 0$ implies $||u_t^n - u_t||_C \to 0$ for any $t \geq 0$.*

Proof of theorem 2. The proof is based on the Gronwall lemma, mean value theorem for the Stieltjes integral, properties of g_d due to the ignoring condition and the Lebesgue-Fatou lemma[38, p.32].

For the simplicity, we first consider a particular case when the generating function $g = g_c \equiv g_{ac} + g_s$ i.e. it does not contain the discrete delays.

One can check (see (8)) that

$$||I_2|| \leq M_b M_f |\Omega|^{\frac{3}{2}} \cdot V_{-r}^0[g_c(\varphi) - g_c(\psi)]. \tag{15}$$

Hence (4), (6), (9), (15) and A4 (see (14)) imply

$$||F_c(\varphi) - F_c(\psi)|| \le L_{F_c}||\varphi - \psi||_C \quad \text{with } L_{F_c} \equiv M_f|\Omega| \left(L_b M_{Vg_c} + M_b|\Omega|^{\frac{1}{2}} L_{Vg_c} \right). \tag{16}$$

Hence

$$||u_t^1 - u_t^2||_C \le ||\varphi - \psi||_C + L_{F_c} \cdot \int_0^t ||u_s^1 - u_s^2||_C \, ds.$$

The last estimate (by the Gronwall lemma) implies

$$||u_t^1 - u_t^2||_C \le e^{L_{F_c} t} \cdot ||\varphi - \psi||_C.$$

That is

$$||u_t^1 - u_t^2||_C \le C_T \cdot ||\varphi - \psi||_C, \quad \forall t \in [0, T], \qquad \text{with } C_T \equiv e^{L_{F_c} T}. \tag{17}$$

We proved the uniqueness of mild solutions and the continuity with respect to initial data in the case $g = g_c$.

The second particular case $g = g_d$ (the purely discrete delay) and only one point of discontinuity has been considered in details in [25]. It was proved in [25] that A5 implies the desired result.

Now we consider the general case (both discrete and continuous delays, including the case of multiple discrete delays). Consider a sequence $\{\varphi^n\} \subset C$ such that $||\varphi^n - \varphi||_C \to 0$ and denote the corresponding mild solutions by $u^n(t) = u^n(t; \varphi^n)$ and $u(t) = u(t; \varphi)$. Using the splitting $F = F_d + F_c$, we have, by definition,

$$u^n(t) - u(t) = e^{-At}(\varphi^n(0) - \varphi(0)) + \int_0^t e^{-A(t-\tau)} \{F_d(u_\tau^n) - F_d(u_\tau)\} \, d\tau$$

$$+ \int_0^t e^{-A(t-\tau)} \{F_c(u_\tau^n) - F_c(u_\tau)\} \, d\tau.$$

Using (16), one gets

$$||u^n(t) - u(t)|| = ||\varphi^n(0) - \varphi(0)| + \int_0^t ||F_d(u_\tau^n) - F_d(u_\tau)|| \, d\tau + L_{F_c} \int_0^t ||u_\tau^n - u_\tau||_C \, d\tau.$$

Hence

$$||u_t^n - u_t||_C = ||\varphi^n - \varphi||_C + \int_0^t ||F_d(u_\tau^n) - F_d(u_\tau)|| \, d\tau + L_{F_c} \int_0^t ||u_\tau^n - u_\tau||_C \, d\tau$$

$$= G^n(t) + L_{F_c} \int_0^t ||u_\tau^n - u_\tau||_C \, d\tau, \tag{18}$$

where $G^n(t) \equiv ||\varphi^n - \varphi||_C + \int_0^t ||F_d(u_\tau^n) - F_d(u_\tau)|| \, d\tau$ is a nondecreasing function.

Multiply the last estimate by $e^{-L_{F_c} t}$ to get

$$\frac{d}{dt} \left(e^{-L_{F_c} t} \int_0^t ||u_\tau^n - u_\tau||_C \, d\tau \right) \le e^{-L_{F_c} t} G^n(t),$$

which, after integration from 0 to t, shows that ($G^n(t)$ is nondecreasing)

$$e^{-L_{F_c}t}\int_0^t ||u^n_\tau - u_\tau||_C\, d\tau \le \int_0^t e^{-L_{F_c}\tau} G^n(\tau)\, d\tau \le G^n(t)\int_0^t e^{-L_{F_c}\tau}\, d\tau = G^n(t)\left(1-e^{-L_{F_c}t}\right) L_{F_c}^{-1}.$$

We have

$$L_{F_c}\int_0^t ||u^n_\tau - u_\tau||_C\, d\tau \le G^n(t)\left(e^{L_{F_c}t}-1\right).$$

We substitute the last estimate into (18) to obtain

$$||u^n_t - u_t||_C \le G^n(t)\cdot e^{L_{F_c}t}. \tag{19}$$

Now our goal is to show that for any fixed $t \in [0, \eta_{ign})$ one has $G^n(t) \to 0$ when $n \to \infty$ (i.e. $||\varphi^n - \varphi||_C \to 0$).

Let us consider the extension functions

$$\overline{\varphi}(s) \equiv \left[\begin{array}{ll} \varphi(s) & s \in [-r, 0]; \\ \varphi(0) & s \in (0, \eta_{ign}) \end{array}\right. \qquad \text{and} \qquad \overline{\varphi}^n(s) \equiv \left[\begin{array}{ll} \varphi^n(s) & s \in [-r, 0]; \\ \varphi^n(0) & s \in (0, \eta_{ign}) \end{array}\right. .$$

As in [25], the *ignoring condition A5* implies that for all $t \in [0, \eta_{ign})$ we have $F_d(u_t) = F_d(\overline{\varphi}_t)$ and $F_d(u^n_t) = F_d(\overline{\varphi}^n_t)$. It is easy to see that the convergence $||\varphi^n - \varphi||_C \to 0$ implies $||\overline{\varphi}^n_\tau - \overline{\varphi}_\tau||_C \to 0$ for any $\tau \in [0, \eta_{ign})$. Hence the continuity of F_d implies $||F_d(\overline{\varphi}^n_\tau) - F_d(\overline{\varphi}_\tau)|| \to 0$ for any $\tau \in [0, \eta_{ign})$. This allows us to use the Lebesgue-Fatou lemma (see [38, p.32]) for the scalar function $||F_d(\overline{\varphi}^n_\tau) - F_d(\overline{\varphi}_\tau)||$ to conclude that $G^n(t) \to 0$ when $n \to \infty$ (for any fixed $t \in [0, \eta_{ign})$). So, we proved the continuity of the mild solutions with respect to initial functions for all $t \in [0, \eta_{ign})$. Particularly, it gives the uniqueness of solutions. For bigger time values we use the chain rule (by the uniqueness) for steps less than or equal to, say $\eta_{ign}/2$. More precisely, we denote by $q \equiv \left[\frac{2t}{\eta_{ign}}\right]$ (here $[\cdot]$ is the integer part of a real number) and write $u(t;\varphi) = u(\underbrace{\eta_{ign}/2; u(\eta_{ign}/2; \ldots}_{q\ times}; u(t - q\cdot\eta_{ign}/2; \varphi)))$. The composition of continuous mappings is continuous. The proof of theorem 2 is complete. ∎

In the standard way we define an evolution semigroup $S_t : C \to C$ by the rule

$$S_t\varphi \equiv u_t,$$

where u is the unique mild solution of (1), (3).

Remark 5. *The continuity of S_t with respect to time variable follows from definition 1 (the solution is a continuous function $u \in C([-r, T]; L^2(\Omega))$). This and the continuity of S_t with respect to initial function (see theorem 2) particularly mean that, under assumptions A1)-A5), the initial value problem (1), (3) is* `well-posed` *in the space C in the sense of J. Hadamard [8, 9].*

The last remark means that the pair (S_t, C) forms the dynamical system (for the definition see e.g [2, 31, 6]).

Following the line of argument given in [25, theorem 2] we show that the dynamical system (S_t, C) generated by initial value problem (1), (3) possesses a compact global attractor (for more details on attractors see, for example [2, 31, 6])).

More precisely, we have the following result.

Theorem 3. *Assume the function $b : \mathbb{R} \to \mathbb{R}$ is a Lipschitz and bounded map ($|b(w)| \le C_b$ for all $w \in \mathbb{R}$) and $f : \Omega - \Omega \to \mathbb{R}$ is a bounded and measurable function ($|f(\cdot)| \le M_f$). Let assumptions A1-A5 be satisfied. Then the dynamical system (S_t, C) has a compact global attractor which is a compact set in all spaces $C_\delta \equiv C([-r, 0]; D(A^\delta)), \forall \delta \in [0, \frac{1}{2})$.*

The proof is based on the classical theorem on the existence of a compact global attractor for a dissipative and asymptotically compact semigroup [2, 31, 6] and technique developed in [25, theorem 2].

As an application we can consider the diffusive Nicholson's blowflies equation (see e.g. [30]) with state-dependent delays, i.e. equation (1) where $-A$ is the Laplace operator with the Dirichlet boundary conditions, $\Omega \subset \mathbb{R}^{n_0}$ is a bounded domain with a smooth boundary, the nonlinear (birth) function b is given by $b(w) = p \cdot we^{-w}$. The function b is bounded, so under assumptions A1-A5, we conclude that the initial value problem (1) and (3) is well-posed in C and the dynamical system (S_t, C) has a compact global attractor (theorem 3).

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